IOP Conference Series: Materials Science and Engineering

PAPER • **OPEN ACCESS**

The problem of plane bending a direct composite beam of arbitrary cross-section and the prerequisites for its approximate analytical solution

To cite this article: S B Koval'chuk *et al* 2021 *IOP Conf. Ser.: Mater. Sci. Eng.* **1164** 012025

IOP Conf. Series: Materials Science and Engineering 1164 (2021) 012025 doi:10.1088/1757-899X/1164/1/012025

The problem of plane bending a direct composite beam of arbitrary cross-section and the prerequisites for its approximate analytical solution

S B Koval'chuk[1,*], O V Goryk[1] and A V Antonets[1]

[1]Poltava State Agrarian Academy, 1/3 H. Skovorody Str., Poltava, 36003, Ukraine

[*]E-mail: stanislav.kovalchuk@pdaa.edu.ua

Abstract. The approach to the reduction of the spatial problem of plane bending a composite discrete-inhomogeneous beam of arbitrary cross-section to the approximate two-dimensional bending problem of the equivalent multilayer beam has been discussed here. The result is represented in the form of relations for determination the characteristics of the equivalent multilayer structure by physical and mechanical materials characteristics of the original beam's phases and the system of static, geometric and physical relations of the corresponding two-dimensional problem. The obtained equations are similar to the plane problem equations of the elasticity theory, but instead of stresses, they contain internal efforts consolidated to the main plane of the beam. The equations of the approximate two-dimensional problem were used to solve the problem of static bending a composite console of arbitrary structure with a load on the free end, taking into account the uniform change of the temperature field. The given system of equations and relations is the starting point for the construction of non-classical deformation models and solving a wide range of problems concerning the deformation of a direct composite beams.

1. Introduction

Beams of discrete-inhomogeneous, composite structure are increasingly used in various fields of mechanical engineering and construction. The research of mechanics deformation of such elements contributes to the development of methods for their design, which is a necessary condition for effective practical implementation of composites. However, the construction of theoretical models of composite beams is complicated by anisotropy of mechanical properties of their material, heterogeneity of the structure, significant susceptibility to transverse displacements, uneven thermal expansion of composite components, even with a uniform temperature field, etc.

For most traditional structural materials, the model of a linear-elastic body is acceptable when solving problems of mechanics. In [1, 2] it was experimentally established that carbon plastics, even at elevated temperatures, shows an almost linear relationship between stresses and strains before failure. Despite the significant physical nonlinearity of the reinforced concrete matrix material [3], for such composites, with some caveats, it is also permissible to assume the linear-elastic work of the components. It has become topical to develop deformation models of composite elements based on the linear theory of elasticity.

The exact solutions of spatial deformation problems of composite beams have been obtained only for simple shapes and cross-sectional structures (circle, annular multilayer structure) [4-7] and in cases of loading (tension-compression, pure bend).

A lot of works [8-35] have suggested exact solutions of plane problems of elasticity theory concerning composite beams, directly or indirectly. Such solutions preferably allow one of the types of external loads to be taken into account. There are also works devoted to bending by force and moment at the end [8-12], uniformly distributed normal load on the longitudinal faces [8, 13-21], linearly distributed load [22-25], load distributed by the law of sinusoids [26, 27]. Several works consider more complex types and combinations of loads. In works [28-31], the load is considered as the sum of the power series, in [32, 33, 35] it is studied as trigonometric series, and in [34] it is discussed as their combinations. Most of these works are devoted to composite homogeneous or continuously inhomogeneous beams with different types of elastic materials symmetry. Only in some works [9, 10, 12, 17, 21, 25, 35] multilayer beams are presented.

Plane solutions for composite beams are relatively simple and give precise distributed components of the stress-strain state. They can be directly used only for the calculation of beams with a rectangular cross-section. Their structure is formed by a package of flat layers perpendicular to the force plane. However, in practice, composite beams have a more complex shape and cross-sections structure.

Non-classical models of bending a composite beams, for example [36, 37, 38, 39], are also constructed as approximate plane problem solutions of the elasticity theory. In some works, while constructing such models, there is a possibility to take into account the arbitrary structure of the cross-sections. In particular, the construction of an iterative model in [38] is carried out by introducing static and kinematic hypotheses at the level of the spatial problem. Next, integrating by the width of the section, the three-dimensional equations of the elasticity theory are reduced to generalized two-dimensional ones, but without taking into account the susceptibility of beam materials to transverse compression. The combination of such an approach with methods for constructing exact solutions of plane problems of the elasticity theory for multilayer beams is quite promising. This will allow to obtain practical ratios for determining stress-strain state when bending composite beams with an arbitrary cross-sectional structure.

2. Materials, hypotheses and research methods

Consider a direct composite beam with a constant structure and cross-sectional dimensions relative to length – see figure 1. The beam consists of m discrete continuous single-connected or multi-connected phases $P_1, P_2, P_3 \ldots, P_k, \ldots, P_m$, which include the matrix, reinforcement, adhesive layers and other elements of the composite, made of appropriate different materials (figure 1, a). The phases of the beam are rigidly connected on common surfaces, there is no relative displacement and separation. The beam has a longitudinal plane of symmetry, relative to which its cross-sections are symmetrical in shape and structure.

The beam is under the action of external loads distributed on its longitudinal faces with a width of $b_{1,2}$ and the ends – see figure 1, b. Normal and tangential loads are distributed symmetrically with respect to the main plane of the beam (xOz). Longitudinal lateral cylindrical surfaces, with generators $\upsilon_{1,2}(z)$ are free from external loadings. The figure 1, b shows the loads acting only in the accepted positive directions. Despite this conditionality, we assume that the load system is balanced.

The material of the composite beam's phases can be homogeneous or continuously inhomogeneous (functionally gradient) in the plane of the cross section. It can also be isotropic or orthotropic with planes of elastic symmetry parallel to the coordinate planes of the coordinate system xyz. Physico-mechanical characteristics of the materials of the beam phases are known:

$$\left\| E_x^{[k]}, E_y^{[k]}, E_z^{[k]}, G_{xz}^{[k]}, G_{zy}^{[k]}, G_{yx}^{[k]}, \nu_{xz}^{[k]}, \nu_{zy}^{[k]}, \nu_{xy}^{[k]}, \vartheta_x^{[k]}, \vartheta_y^{[k]}, \vartheta_z^{[k]}, \rho^{[k]} \right\| = \left\| S_a^{[k]} \right\|,$$

where $S_a^{[k]} = \text{const}$ – within a homogeneous phase and $S_a^{[k]} = f(y,z)$ – in the general case of a continuous-inhomogeneous phase.

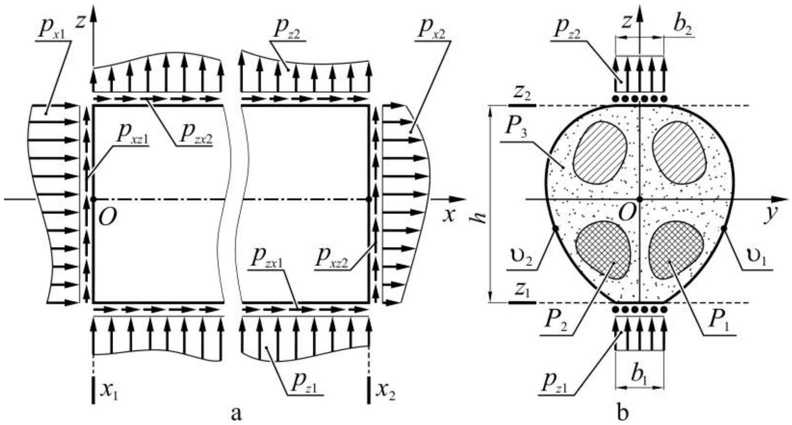

Figure 1. Scheme of a composite beam: (a) side view; (b) cross section, right view.

For the whole beam, the characteristics of the inhomogeneous material will be piecewise continuous functions $\mu_a^S = \mu_a^S(y, z)$ with discontinuities of the first kind at the boundaries of the phases, which can be conditionally represented in the following form:

$$\mu_a^S = \sum_{k=1}^{m} \left(p_k S_a^{[k]} \right), \tag{1}$$

where p_k – characteristic function of phase k :

$$p_k = p_k\left(K'(y,z)\right) = \begin{cases} 1, & K' \in P_k; \\ 0, & K' \notin P_k. \end{cases} \tag{2}$$

For the analytical description of the characteristic functions (2), the Heaviside function can be used. For example, for an arbitrary phase of a composite beam, which is limited in cross-section on the sides by continuous curves $\upsilon_{k,l-1}(z)$ and $\upsilon_{k,l}(z)$, and at the bottom and top by horizontal lines z_{k-1} and z_k, the characteristic function can be written as follows:

$$p_k = \left[H\left(y - \upsilon_{k,l-1}\right) - H\left(y - \upsilon_{k,l}\right) \right] \left[H\left(z - z_{k-1}\right) - H\left(z - z_k\right) \right]. \tag{3}$$

For a beam with an arbitrary cross-sectional structure, it is always possible to divide it in height by z_k horizontal lines into a number of generalized layers. Within these layers, the characteristic functions of the phases can be represented as (2). This gives a generalized approach to the analytical representation of the mechanical characteristics functions of an inhomogeneous beam (3).

Due to the accepted condition of absolutely rigid contact of phases, the material of the considered beam is continuous. Accordingly, the classical equilibrium equations and geometric relations of the elasticity theory are valid for such an element:

$$\begin{gathered} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{yx}}{\partial y} + \mu^{\rho}\left(F_x^V - \frac{\partial^2 u}{\partial t^2} \right) = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \mu^{\rho}\left(F_y^V - \frac{\partial^2 v}{\partial t^2} \right) = 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \mu^{\rho}\left(F_z^V - \frac{\partial^2 w}{\partial t^2} \right) = 0, \end{gathered} \tag{4}$$

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x}, && \varepsilon_z = \frac{\partial w}{\partial z}, && \varepsilon_y = \frac{\partial v}{\partial y}, \\ &\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, && \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, && \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \end{aligned} \tag{5}$$

where F_x^V, F_y^V, F_z^V – projections of bulk forces per unit mass of beam material; μ^ρ – function of density distribution of beam materials; t – time.

The physical dependences of Hooke's law also remain valid within individual phases. However, for all inhomogeneous beams, they must take into account the change in mechanical characteristics in its cross-section. This can be achieved by replacing the elastic constants in the classical equations with the distribution functions of physical and mechanical characteristics:

$$\begin{aligned} &\varepsilon_x = \frac{\sigma_x}{\mu_x^E} - \frac{\mu_{yx}^\nu \sigma_y}{\mu_y^E} - \frac{\mu_{zx}^\nu \sigma_z}{\mu_z^E} + \mu_x^\vartheta \Delta T, && \gamma_{xz} = \frac{1}{\mu_{xz}^G} \tau_{xz}, \\ &\varepsilon_y = \frac{\sigma_y}{\mu_y^E} - \frac{\mu_{xy}^\nu \sigma_x}{\mu_x^E} - \frac{\mu_{zy}^\nu \sigma_z}{\mu_z^E} + \mu_y^\vartheta \Delta T, && \gamma_{zy} = \frac{1}{\mu_{zy}^G} \tau_{zy}, \\ &\varepsilon_z = \frac{\sigma_z}{\mu_z^E} - \frac{\mu_{yz}^\nu \sigma_y}{\mu_y^E} - \frac{\mu_{xz}^\nu \sigma_x}{\mu_x^E} + \mu_z^\vartheta \Delta T, && \gamma_{yx} = \frac{1}{\mu_{yx}^G} \tau_{yx}, \end{aligned} \tag{6}$$

where ΔT – change in temperature field inside the beam.

Given the received load (figure 1) static boundary conditions on the longitudinal faces and ends of the beam will be written as

$$\sigma_x |_{x=x_\varsigma} = (-1)^\varsigma p_{x,\varsigma}, \qquad \tau_{xz} |_{x=x_\varsigma} = (-1)^\varsigma p_{xz,\varsigma}, \qquad \tau_{xy} |_{x=x_\varsigma} = 0, \tag{7}$$

$$\tau_{zx} |_{z=z_\varsigma} = (-1)^\varsigma p_{zx,\varsigma}, \qquad \sigma_z |_{z=z_\varsigma} = (-1)^\varsigma p_{z,\varsigma}, \qquad \tau_{zy} |_{z=z_\varsigma} = 0, \ \varsigma = 1, 2. \tag{8}$$

For a symmetrical section, the side curves of its contour are distinguished only by a sign:

$$\upsilon_2(z) = -\upsilon_1(z) = \upsilon(z), \tag{9}$$

where $\upsilon(z)$ – an arbitrary piecewise continuous function that does not change the sign when $z \in (z_1, z_2)$.

Then, based on geometric considerations, we write the guide cosines of the normal to the lateral cylindrical surfaces:

$$m_y = (-1)^\varsigma \left(1 + \left(\frac{d\upsilon}{dz}\right)^2\right)^{-1/2}, \qquad n_z = -\frac{d\upsilon}{dz} \left(1 + \left(\frac{d\upsilon}{dz}\right)^2\right)^{-1/2}, \ \varsigma = 1, 2. \tag{10}$$

Given the absence of loads on the side surfaces, the static boundary conditions for them, taking into account (10), will be written as

$$\left(\tau_{yx} + (-1)^{\varsigma+1} \tau_{zx} \frac{d\upsilon}{dz}\right)\Big|_{y=(-1)^\varsigma \upsilon} = 0, \ \left(\tau_{yz} + (-1)^{\varsigma+1} \sigma_z \frac{d\upsilon}{dz}\right)\Big|_{y=(-1)^\varsigma \upsilon} = 0, \ \left(\sigma_y + (-1)^{\varsigma+1} \tau_{yz} \frac{d\upsilon}{dz}\right)\Big|_{y=(-1)^\varsigma \upsilon} = 0. \tag{11}$$

Equations (4),(5) and (6) constitute a complete system of equations of the elasticity theory spatial problem for the considered composite beam. Their analytical solution in general in accordance with the boundary conditions (7), (8), (11) encounters significant mathematical difficulties.

However, by introducing some physical assumptions about the properties of phase materials, the dimension of such a problem can be reduced, which simplifies its research and analytical solution.

3. Reduction of the spatial problem of plane bending a composite discrete-inhomogeneous beam of arbitrary cross-section to the approximate two-dimensional problem

We consider the beam narrow enough to neglect the deformation of the composite's phases along the axis Oy. This is equivalent to the assumption of absolute stiffness of the materials of all composite beam's phases in the specified direction:

$$\mu_{yx}^{\nu}, \mu_{xy}^{\nu}, \mu_{yz}^{\nu}, \mu_{zy}^{\nu}, \mu_{y}^{\vartheta} = 0, \qquad \mu_{y}^{E}, \mu_{zy}^{G}, \mu_{yx}^{G} \to \infty. \tag{12}$$

We also assume a uniform distribution of the temperature field along the axis Oy:

$$\Delta T = \Delta T(\eta, \xi, t). \tag{13}$$

Substituting (12) for the physical dependences (6), we have:

$$\begin{gathered} \varepsilon_x = \frac{\sigma_x}{\mu_x^E} - \frac{\mu_{zx}^{\nu}\sigma_z}{\mu_z^E} + \mu_x^{\vartheta}\Delta T, \quad \varepsilon_z = \frac{\sigma_z}{\mu_z^E} - \frac{\mu_{xz}^{\nu}\sigma_x}{\mu_x^E} + \mu_z^{\vartheta}\Delta T, \quad \varepsilon_y = 0, \\ \gamma_{xz} = \frac{1}{\mu_{xz}^G}\tau_{xz}, \qquad \gamma_{zy} = 0, \qquad \gamma_{yx} = 0. \end{gathered} \tag{14}$$

Taking into account (14), the geometric relations (5) have the form:

$$\begin{gathered} \varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \qquad 0 = \frac{\partial v}{\partial y}, \\ \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad 0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad 0 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \end{gathered} \tag{15}$$

Solving the third, fifth and sixth relations (15), taking into account the assumption of plane deformation of the beam, we obtain:

$$\begin{gathered} v = 0, \qquad u = u(x,z,t), \qquad w = w(x,z,t), \\ \varepsilon_x = \varepsilon_x(x,z,t), \qquad \varepsilon_z = \varepsilon_z(x,z,t), \qquad \gamma_{xz} = \gamma_{xz}(x,z,t). \end{gathered} \tag{16}$$

Solving (14) with respect to stresses and integrating the width of the section (within $y = -\upsilon(z)\ldots\upsilon(z)$), taking into account (13) and (16) we have:

$$\bar{\sigma}_x = \bar{\mu}_{xx}^S\varepsilon_x + \bar{\mu}_{zx}^S\varepsilon_z + \bar{\mu}_x^{\beta}\Delta T, \qquad \bar{\sigma}_z = \bar{\mu}_{zz}^S\varepsilon_z + \bar{\mu}_{xz}^S\varepsilon_x + \bar{\mu}_z^{\beta}\Delta T, \qquad \bar{\tau}_{xz} = \gamma_{xz}\bar{\mu}_{xz}^G, \tag{17}$$

where $\bar{\mu}_{xx}^S, \ldots, \bar{\mu}_{xz}^G$ – functions of the consolidated elastic characteristics of the inhomogeneous material of the composite beam:

$$\begin{gathered} \bar{\mu}_{xx}^S = \int_{-\upsilon}^{\upsilon} \left\{ \mu_x^E \mu_x^E \left[\mu_x^E - \mu_z^E \left(\mu_{xz}^{\nu}\right)^2 \right]^{-1} \right\} dy, \qquad \bar{\mu}_{zz}^S = \int_{-\upsilon}^{\upsilon} \left\{ \mu_x^E \mu_z^E \left[\mu_x^E - \mu_z^E \left(\mu_{xz}^{\nu}\right)^2 \right]^{-1} \right\} dy, \\ \bar{\mu}_{xz}^S = \bar{\mu}_{zx}^S = \int_{-\upsilon}^{\upsilon} \left\{ \mu_x^E \mu_z^E \mu_{xz}^{\nu} \left[\mu_x^E - \mu_z^E \left(\mu_{xz}^{\nu}\right)^2 \right]^{-1} \right\} dy, \quad \bar{\mu}_{xz}^G = \int_{-\upsilon}^{\upsilon} \mu_{xz}^G dy, \\ \bar{\mu}_x^{\beta} = -\int_{-\upsilon}^{\upsilon} \left\{ \mu_x^E \frac{\mu_x^E \mu_x^{\vartheta} + \mu_z^E \mu_{xz}^{\nu} \mu_z^{\vartheta}}{\mu_x^E - \mu_z^E \left(\mu_{xz}^{\nu}\right)^2} \right\} dy, \qquad \bar{\mu}_z^{\beta} = -\int_{-\upsilon}^{\upsilon} \left\{ \mu_z^E \frac{\mu_x^E \mu_z^{\vartheta} + \mu_x^E \mu_{xz}^{\nu} \mu_x^{\vartheta}}{\mu_x^E - \mu_z^E \left(\mu_{xz}^{\nu}\right)^2} \right\} dy; \end{gathered} \tag{18}$$

$\bar{\sigma}_x,\ldots,\bar{\tau}_{zy}$ – functions of the consolidated stresses along the width of the beam's cross-section:

$$\bar{\sigma}_x=\int_{-\upsilon}^{\upsilon}\sigma_x dy,\ \bar{\sigma}_z=\int_{-\upsilon}^{\upsilon}\sigma_z dy,\ \bar{\sigma}_y=\int_{-\upsilon}^{\upsilon}\sigma_y dy,\ \bar{\tau}_{zx}=\int_{-\upsilon}^{\upsilon}\tau_{zx} dy. \tag{19}$$

In equation (18) it is taken into account that for a composite beam with isotropic and orthotropic phases: $\mu_x^E\mu_{zx}^{\nu}=\mu_z^E\mu_{xz}^{\nu}$, as a result we have $\bar{\mu}_{zx}^S=\bar{\mu}_{xz}^S$.

Solving (17) with respect to deformations and comparing with the corresponding expressions (14) we obtain the summary relations of Hooke's law for a composite beam:

$$\varepsilon_x=\frac{\bar{\sigma}_x}{\bar{\mu}_x^E}-\frac{\bar{\mu}_{xz}^{\nu}}{\bar{\mu}_x^E}\bar{\sigma}_z+\bar{\mu}_x^{\vartheta}\Delta T,\ \ \varepsilon_z=\frac{\bar{\sigma}_z}{\bar{\mu}_z^E}-\frac{\bar{\mu}_{xz}^{\nu}}{\bar{\mu}_x^E}\bar{\sigma}_x+\bar{\mu}_z^{\vartheta}\Delta T,\ \ \gamma_{xz}=\frac{1}{\bar{\mu}_{xz}^G}\bar{\tau}_{xz}. \tag{20}$$

and ratios for consolidated mechanical characteristics:

$$\begin{aligned}
&\bar{\mu}_x^E=\left(\bar{\mu}_{zz}^S\right)^{-1}\left(\bar{\mu}_{xx}^S\bar{\mu}_{zz}^S-\bar{\mu}_{xz}^S\bar{\mu}_{zx}^S\right),\quad \bar{\mu}_{xz}^{\nu}=\left(\bar{\mu}_{zz}^S\right)^{-1}\bar{\mu}_{zx}^S,\quad \bar{\mu}_x^{\vartheta}=\left(\bar{\mu}_{xx}^S\bar{\mu}_{zz}^S-\bar{\mu}_{xz}^S\bar{\mu}_{zx}^S\right)^{-1}\left(\bar{\mu}_{zx}^S\bar{\mu}_z^{\beta}-\bar{\mu}_{zz}^S\bar{\mu}_x^{\beta}\right),\\
&\bar{\mu}_z^E=\left(\bar{\mu}_{xx}^S\right)^{-1}\left(\bar{\mu}_{xx}^S\bar{\mu}_{zz}^S-\bar{\mu}_{xz}^S\bar{\mu}_{zx}^S\right),\qquad\qquad\qquad\qquad \bar{\mu}_z^{\vartheta}=\left(\bar{\mu}_{xx}^S\bar{\mu}_{zz}^S-\bar{\mu}_{xz}^S\bar{\mu}_{zx}^S\right)^{-1}\left(\bar{\mu}_{xz}^S\bar{\mu}_x^{\beta}-\bar{\mu}_{xx}^S\bar{\mu}_z^{\beta}\right).
\end{aligned} \tag{21}$$

We equate the right-hand sides of the corresponding relations (14), (20) and solve the obtained equations with respect to stresses $\sigma_x,\sigma_z,\tau_{xz}$. As a result, we obtain the dependences for the transition from the consolidated stresses $\bar{\sigma}_x,\bar{\sigma}_z,\bar{\tau}_{xz}$ to distributed across the width of the cross-section:

$$\begin{aligned}
\sigma_x&=\frac{\mu_x^E}{\mu_x^E-\mu_z^E\left(\mu_{xz}^{\nu}\right)^2}\frac{1}{\bar{\mu}_x^E\bar{\mu}_z^E}\left[\begin{array}{r}\left(\mu_x^E-\mu_z^E\mu_{xz}^{\nu}\bar{\mu}_{xz}^{\nu}\right)\bar{\mu}_z^E\bar{\sigma}_x+\left(\mu_z^E\mu_{xz}^{\nu}\bar{\mu}_x^E-\mu_x^E\bar{\mu}_{xz}^{\nu}\bar{\mu}_z^E\right)\bar{\sigma}_z+\\ +\left[\mu_z^E\mu_{xz}^{\nu}\left(\bar{\mu}_z^{\vartheta}-\mu_z^{\vartheta}\right)+\mu_x^E\left(\bar{\mu}_x^{\vartheta}-\mu_x^{\vartheta}\right)\right]\bar{\mu}_x^E\bar{\mu}_z^E\Delta T\end{array}\right],\\
\sigma_z&=\frac{\mu_x^E\mu_z^E}{\mu_x^E-\mu_z^E\left(\mu_{xz}^{\nu}\right)^2}\frac{1}{\bar{\mu}_x^E\bar{\mu}_z^E}\left[\begin{array}{r}\left(\mu_{xz}^{\nu}-\bar{\mu}_{xz}^{\nu}\right)\bar{\mu}_z^E\bar{\sigma}_x+\left(\bar{\mu}_x^E-\mu_{xz}^{\nu}\bar{\mu}_z^E\bar{\mu}_{xz}^{\nu}\right)\bar{\sigma}_z+\\ +\left[\mu_{xz}^{\nu}\left(\bar{\mu}_x^{\vartheta}-\mu_x^{\vartheta}\right)+\left(\bar{\mu}_z^{\vartheta}-\mu_z^{\vartheta}\right)\right]\bar{\mu}_x^E\bar{\mu}_z^E\Delta T\end{array}\right],\\
\tau_{xz}&=\frac{\mu_{xz}^G}{\bar{\mu}_{xz}^G}\bar{\tau}_{xz}.
\end{aligned} \tag{22}$$

Integrate the equilibrium equation (4) along the width of the beam's cross section (within $y=-\upsilon(z)\ldots\upsilon(z)$). Taking into account the rule of the definite integral differentiation respect to the parameter and equations (9), (16), (19), we obtain:

$$\begin{aligned}
&\frac{\partial\bar{\sigma}_x}{\partial x}+\frac{\partial\bar{\tau}_{zx}}{\partial z}+\bar{F}_x^V-\bar{\mu}^{\rho}\frac{\partial^2 u}{\partial t^2}=-\left(\tau_{yx}-\frac{d\upsilon}{dz}\tau_{zx}\right)\Big|_{y=\upsilon}+\left(\tau_{yx}+\frac{d\upsilon}{dz}\tau_{zx}\right)\Big|_{y=-\upsilon},\\
&\frac{\partial\bar{\tau}_{xy}}{\partial x}+\frac{\partial\bar{\tau}_{zy}}{\partial z}+\bar{F}_y^V-\bar{\mu}^{\rho}\frac{\partial^2 v}{\partial t^2}=-\left(\sigma_y-\frac{d\upsilon}{dz}\tau_{zy}\right)\Big|_{y=\upsilon}+\left(\sigma_y+\frac{d\upsilon}{dz}\tau_{zy}\right)\Big|_{y=-\upsilon},\\
&\frac{\partial\bar{\tau}_{zx}}{\partial x}+\frac{\partial\bar{\sigma}_z}{\partial z}+\bar{F}_z^V-\bar{\mu}^{\rho}\frac{\partial^2 w}{\partial t^2}=-\left(\tau_{yz}-\frac{d\upsilon}{dz}\sigma_z\right)\Big|_{y=\upsilon}+\left(\tau_{yz}+\frac{d\upsilon}{dz}\sigma_z\right)\Big|_{y=-\upsilon},
\end{aligned} \tag{23}$$

where $\bar{\tau}_{xy},\bar{\tau}_{zy}$ – functions of the consolidated stresses along the width of the beam's csection (similarly (22)); $\bar{F}_x^V$, $\bar{F}_y^V$, $\bar{F}_z^V$, $\bar{\mu}^{\rho}$ – summary components of bulk forces and density distribution function:

$$\bar{F}_x^V = \int_{-\upsilon}^{\upsilon} \left(\mu^\rho F_x^V\right) dy,\ \bar{F}_y^V = \int_{-\upsilon}^{\upsilon} \left(\mu^\rho F_y^V\right) dy,\ \bar{F}_z^V = \int_{-\upsilon}^{\upsilon} \left(\mu^\rho F_z^V\right) dy,\ \bar{\mu}^\rho = \int_{-\upsilon}^{\upsilon} \mu^\rho dy$$

The right-hand sides of equations (23) are similar to the corresponding boundary conditions (11), taking into account which we have:

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{zx}}{\partial z} + \bar{F}_x^V - \bar{\mu}^\rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{zy}}{\partial z} + \bar{F}_y^V - \bar{\mu}^\rho \frac{\partial^2 v}{\partial t^2} = 0,$$
$$\frac{\partial \bar{\tau}_{zx}}{\partial x} + \frac{\partial \bar{\sigma}_z}{\partial z} + \bar{F}_z^V - \bar{\mu}^\rho \frac{\partial^2 w}{\partial t^2} = 0. \tag{24}$$

In the case where the load on the lateral cylindrical surfaces $p_{x,\varsigma}, p_{y,\varsigma}, p_{z,\varsigma} \neq 0$, but symmetrical: $p_{x,\varsigma} = p_x^\wedge$, $p_{z,\varsigma} = p_z^\wedge$, $p_{y,\varsigma} = (-1)^\varsigma p_y^\wedge$, in the right parts of the first and third equations (23) will be the sums of projections of the corresponding external loads on the plane of the beam symmetry (xOz),and the second equation will be zero, as in the absence of loads:

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{zx}}{\partial z} + \bar{F}_x^V - \bar{\mu}^\rho \frac{\partial^2 u}{\partial t^2} = -2 p_x^\wedge \sqrt{1 + \left(\frac{d\upsilon}{dz}\right)^2}, \quad \frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{zy}}{\partial z} + \bar{F}_y^V - \bar{\mu}^\rho \frac{\partial^2 v}{\partial t^2} = 0,$$
$$\frac{\partial \bar{\tau}_{zx}}{\partial x} + \frac{\partial \bar{\sigma}_z}{\partial z} + \bar{F}_z^V - \bar{\mu}^\rho \frac{\partial^2 w}{\partial t^2} = -2 p_z^\wedge \sqrt{1 + \left(\frac{d\upsilon}{dz}\right)^2}.$$

That is, the loads on the lateral cylindrical surfaces will directly enter the system of consolidated static equations and will affect the stress distribution along the cross-section height.

Integrating the static conditions (7) and (8) over the width of the section, we write:

$$\bar{\sigma}_x |_{x=x_\varsigma} = (-1)^\varsigma q_{x,\varsigma}, \qquad \bar{\tau}_{xz} |_{x=x_\varsigma} = (-1)^\varsigma q_{xz,\varsigma}, \qquad \bar{\tau}_{xy} |_{x=x_\varsigma} = 0, \tag{25}$$

$$\bar{\tau}_{zx} |_{z=z_\varsigma} = (-1)^\varsigma q_{zx,\varsigma}, \qquad \bar{\sigma}_z |_{z=z_\varsigma} = (-1)^\varsigma q_{z,\varsigma}, \qquad \bar{\tau}_{zy} |_{z=z_\varsigma} = 0, \ \ \varsigma = 1,2. \tag{26}$$

The second equation (24) and the boundary conditions (25) and (26) will be satisfied if

$$\bar{\tau}_{xy} = 0, \qquad \bar{\tau}_{zy} = 0, \tag{27}$$

нowever, this does not mean that the stresses τ_{xy}, τ_{zy} are zero.

Thus, for a consolidated two-dimensional problem based on (15), (16), (20), (24), (27) we can write the following system of equations:

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{zx}}{\partial z} + \bar{F}_x^V - \bar{\mu}^\rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial \bar{\tau}_{zx}}{\partial x} + \frac{\partial \bar{\sigma}_z}{\partial z} + \bar{F}_z^V - \bar{\mu}^\rho \frac{\partial^2 w}{\partial t^2} = 0, \tag{28}$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \tag{29}$$

$$\varepsilon_x = \frac{\bar{\sigma}_x}{\bar{\mu}_x^E} - \frac{\bar{\mu}_{xz}^\nu}{\bar{\mu}_x^E} \bar{\sigma}_z + \bar{\mu}_x^9 \Delta T, \ \ \varepsilon_z = \frac{\bar{\sigma}_z}{\bar{\mu}_z^E} - \frac{\bar{\mu}_{xz}^\nu}{\bar{\mu}_x^E} \bar{\sigma}_x + \bar{\mu}_z^9 \Delta T, \ \ \gamma_{xz} = \frac{1}{\bar{\mu}_{xz}^G} \bar{\tau}_{xz}. \tag{30}$$

The system of eight equations (28-30) contains 8 unknown functions: $\bar{\sigma}_x$, $\bar{\sigma}_z$, $\bar{\tau}_{zx}$ – consolidated internal efforts (dimension N/m); ε_x, ε_z, γ_{xz} – averaged linear and angular deformations; u, w –

averaged displacements. Efforts $\bar{\sigma}_x$, $\bar{\sigma}_z$, $\bar{\tau}_{zx}$, in addition to the specified system of equations, must meet the summary boundary conditions (25), (26).

In an arbitrary section of the beam, the integral conditions of equilibrium of the beam part, cut off by the cross-section with the coordinate x, must be fulfilled:

$$\int_{z_1}^{z_2} \bar{\sigma}_x dz = -\sum_{\varsigma=1}^{2} \int_0^x q_{zx,\varsigma}|_{x=\theta} \, d\theta + N_1, \qquad \int_{z_1}^{z_2} \bar{\tau}_{xz} dz = -\sum_{\varsigma=1}^{2} \int_0^\eta q_{z,\varsigma}|_{\eta=\theta} \, d\theta + Q_1,$$
$$\int_{z_1}^{z_2} (\bar{\sigma}_x z) dz = -\sum_{\varsigma=1}^{2} \int_0^x \left[(x-\theta) q_{z,\varsigma}|_{x=\theta} + z_\varsigma q_{zx,\varsigma}|_{x=\theta} \right] d\theta + Q_1 x + M_1, \tag{31}$$

where N_1, Q_1, M_1 – components of internal force factors from loads in the initial section:

$$N_1 = -\int_{z_1}^{z_2} q_{x1} dz, \; Q_1 = -\int_{z_1}^{z_2} q_{xz1} dz, \; M_1 = -\int_{z_1}^{z_2} (z q_{x1}) dz. \tag{32}$$

The above approach actually reduces the problem of bending a composite beam with an inhomogeneous cross-section of arbitrary structure to the problem of bending some equivalent rectangular multilayer section of unit width. In the case where the contours of the homogeneous phases of the composite beam consist of vertical and horizontal lines, the generalized layers of such an equivalent beam will have constant physical and mechanical characteristics throughout the thickness. Otherwise, the generalized layers of the equivalent beam will be constantly inhomogeneous or functionally gradient in thickness.

Let the structure of an equivalent multilayer beam consist of m generalized layers under conditions $z = z_{bd1}, z_{bd2}, z_{bd3}, \ldots, z_{bd,k}, \ldots, z_{bd,m}$. Then, at the boundaries of the layers, based on the conditions of absolutely rigid connection of the original beam's phases, which provide equality of full effort and displacement at each point of the common boundary, the following conditions must be met:

$$\bar{\tau}_{xz}^{[k]}|_{z=z_{bd,k}} = \bar{\tau}_{xz}^{[k+1]}|_{z=z_{bd,k}}, \qquad \bar{\sigma}_z^{[k]}|_{z=z_{bd,k}} = \bar{\sigma}_z^{[k+1]}|_{z=z_{bd,k}}, \tag{33}$$

$$w^{[k]}|_{z=z_{bd,k}} = w^{[k+1]}|_{z=z_{bd,k}}, \qquad u^{[k]}|_{z=z_{bd,k}} = u^{[k+1]}|_{z=z_{bd,k}}, \; k = \overline{1, m-1}. \tag{34}$$

The system of equations (28-30) is similar in form to the equations of the plane problem of the elasticity theory, but instead of stresses in static and physical dependences it contains their equivalent. Using this similarity, the solution of this system can be obtained by analogy with the known solutions of the plane problem of the elasticity theory for multilayer beams.

4. Solution of the problem of static bending a composite console of arbitrary structure with a load on the free end taking into account the uniform change of the temperature field

As an example of solving the obtained system of equations, consider the solution of the problem of the composite console's static bending (figure 2) with an arbitrary load on the initial free end ($q_{z,\varsigma}, q_{zx,\varsigma} = 0$), a uniform change in temperature field ($\Delta T = \text{const}$) and in the absence of bulk forces.

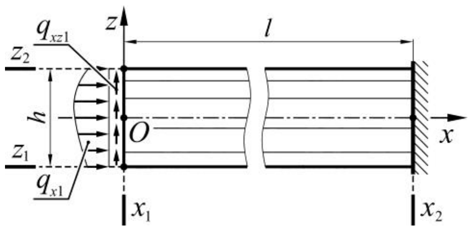

Figure 2. Scheme of a composite beam.

Similarly to [12], where a similar problem for a multilayer beam is considered, but without taking into account thermal deformations, we solve the solution of the problem with respect to the function of tangential stresses $\bar{\tau}_{xz}$. According to the second integral condition (31):

$$\bar{\tau}_{xz} = \bar{\tau}_{xz}\left(z\right). \tag{35}$$

According to the construction of the solution in [21] by successive solution of static, physical and geometric equations, we obtain

$$\bar{\sigma}_x = \bar{\mu}_x^E\left(\frac{N_1+N^T}{\bar{B}_0} - \frac{Q_1}{\bar{B}_2}zx - \frac{M_1+M^T}{\bar{B}_2}z - \bar{\mu}_x^9\Delta T\right),\ \ \sigma_z = 0,\ \ \bar{\tau}_{xz} = \frac{Q_1}{\bar{B}_2}\int_{z_1}^{z}\left(\bar{\mu}_x^E z\right)dz, \tag{36}$$

$$\varepsilon_x = -\frac{Q_1}{\bar{B}_2}zx - \frac{M_1+M^T}{\bar{B}_2}z + \frac{N_1+N^T}{\bar{B}_0},$$
$$\varepsilon_z = \frac{Q_1}{\bar{B}_2}\bar{\mu}_{xz}^{\nu}zx + \frac{M_1+M^T}{\bar{B}_2}\bar{\mu}_{xz}^{\nu}z - \frac{N_1+N^T}{\bar{B}_0} + \left(\bar{\mu}_{xz}^{\nu}\bar{\mu}_x^9 + \bar{\mu}_z^9\right)\Delta T,\ \ \gamma_{xz} = \frac{Q_1}{\bar{B}_2}\frac{1}{\bar{\mu}_{xz}^G}\int_{z_1}^{z}\left(\bar{\mu}_x^E z\right)dz, \tag{37}$$

$$u = -\frac{Q_1}{\bar{B}_2}\left[\frac{x^2 z}{2} - \int_{z_1}^{z}\left(\frac{1}{\bar{\mu}_{xz}^G}\int_{z_1}^{z}\left(\bar{\mu}_x^E z\right)dz - \int_{z_1}^{z}\left(\bar{\mu}_{xz}^{\nu}z\right)dz\right)dz + \bar{D}_2\frac{z-z_1}{h}\right] +$$
$$+\frac{N_1+N^T}{\bar{B}_0}x - \frac{M_1+M^T}{\bar{B}_2}zx + u\,|_{x=0,z=z_2}\frac{z-z_1}{h} + u\,|_{\eta=0,z=z_1}\frac{z_2-z}{h}, \tag{38}$$

$$w = \frac{Q_1}{\bar{B}_2}\left[\frac{x^3}{6} + x\left(\int_{z_1}^{z}\left(\bar{\mu}_{xz}^{\nu}z\right)dz + \frac{\bar{D}_2}{h}\right)\right] + \frac{M_1+M^T}{\bar{B}_2}\left[\frac{x^2}{2} + \int_{z_1}^{z}\left(\mu_{xz}^{\nu}z\right)dz\right] -$$
$$-\frac{N_1+N^T}{\bar{B}_0}\int_{z_1}^{z}\bar{\mu}_{xz}^{\nu}dz + \Delta T\int_{z_1}^{z}\left(\bar{\mu}_{xz}^{\nu}\bar{\mu}_x^9 + \bar{\mu}_z^9\right)dz + u\,|_{x=0,z=z_1}\frac{x}{h} - u\,|_{x=0,z=z_2}\frac{x}{h} + w\,|_{z=z_1,x=0}, \tag{39}$$

where N^T, M^T – internal force and moment from thermal stresses

$$N^T = \Delta T\int_{z_1}^{z_2}\left(\bar{\mu}_x^E\bar{\mu}_x^9\right)dz,\ \ M^T = \Delta T\int_{z_1}^{z_2}\left(\bar{\mu}_x^E\bar{\mu}_x^9 z\right)dz; \tag{40}$$

$\bar{B}_0$, $\bar{B}_2$, $\bar{D}_2$ – integral characteristics of rigidity of a beam cross-section

$$\bar{B}_0 = \int_{z_1}^{z_2}\bar{\mu}_x^E dz,\ \ \bar{B}_2 = \int_{z_1}^{z_2}\int_{z_1}^{z}\left(\bar{\mu}_x^E z\right)dzdz,\ \ \bar{D}_2 = \int_{z_1}^{z_2}\left(\frac{1}{\bar{\mu}_{xz}^G}\int_{z_1}^{z}\left(\bar{\mu}_x^E z\right)dz - \int_{z_1}^{z}\left(\bar{\mu}_{xz}^{\nu}z\right)dz\right)dz; \tag{41}$$

$u\,|_{x=0,z=z_2}$, $u\,|_{\eta=0,z=z_1}$, $w\,|_{z=z_1,x=0}$ – unknown movements of the extreme points the initial end of the beam.

Solution (36-39) is obtained under the condition that the distance z'_O from the lower longitudinal surface to the axis of the beam (Ox) is determined by the ratio

$$z'_O = \int_0^h\left(\bar{\mu}'^E_x z\right)dz \Big/ \int_0^h\bar{\mu}'^E_x dz, \tag{42}$$

where $\bar{\mu}'^E_x$ – distribution function of the longitudinal elastic modulus for the case when the beginning of the coordinate system is on the bottom line of the section.

It should be noted that for a narrow multilayer beam considered in [12], the solution (36-39) will be accurate, although it gives values consolidated by the width of the cross-section values of internal efforts instead of stresses. For a beam of arbitrary structure, these relations constitute an approximate solution of the problem, and do not give all the components of the stress state. However, having the distribution of the consolidated efforts (36), according to (22) we can find an approximate distribution of stresses in the output beam. Using these relations according to the spatial equations of equilibrium (4), we can obtain an approximate distribution of stresses $\tau_{\eta y}, \tau_{\xi y}, \sigma_y$ that will correspond to the physical assumption (12).

Rigid fixing of the end face of a beam with a coordinate $x = x_2$, it is possible to model conditions:

$$u\,|_{x=l,z=z_1} = 0,\;\; u\,|_{x=l,z=z_2} = 0,\;\; w|_{z=z_1,x=l} = 0. \tag{43}$$

Applying (38) and (39) under conditions (43) and solving the obtained equations with respect to the unknown displacements of the initial end, we obtain

$$u\,|_{\eta=0,z=z_1} = \frac{Q_1}{\bar{B}_2}\frac{l^2 z_1}{2} - \frac{N_1 + N^T}{\bar{B}_0} l + \frac{M_1 + M^T}{\bar{B}_2} l z_1,\;\; u\,|_{x=0,z=z_2} = \frac{Q_1}{\bar{B}_2}\frac{l^2 z_2}{2} - \frac{N_1 + N^T}{\bar{B}_0} l + \frac{M_1 + M^T}{\bar{B}_2} l z_2,$$
$$w\,|_{z=z_1,x=0} = \frac{Q_1}{\bar{B}_2}\left(\frac{l^3}{3} - \frac{l\bar{D}_2}{h}\right) + \frac{M_1 + M^T}{\bar{B}_2}\frac{l^2}{2}. \tag{44}$$

Substitution of expressions (44) to relations (38) and (39) gives the completed functions of distribution longitudinal and cross movements of the composite console.

The last expression (44) gives the transverse movement of the lower fiber in the initial cross-section of the console, which is close in value to its deflection arrow:

$$f \approx w\,|_{z=z_1,x=0} = \left(\frac{Q_1 l^3}{3\bar{B}_2} + \frac{M_1 l^2}{2\bar{B}_2}\right) - \frac{Q_1 l\bar{D}_2}{h\bar{B}_2} + \frac{M^T l^2}{2\bar{B}_2}.$$

Analyzing this expression, we can clearly see the components of the pure bending of the console, corresponding to the plane sections hypothesis, the component of the transverse shear deformation and compression, as well as the component of thermal stresses due to uneven expansion the phases of composite.

5. Conclusion

The paper proposes some approach to the reduction of the plane bending spatial problem of composite discrete-inhomogeneous beam of arbitrary cross-section to the approximate two-dimensional problem. This approach is based on the assumption of absolute stiffness of the phases along the width of the beam.This assumption allows us to move from the original spatial problem to the equivalent plane problem of bending a multilayer beam of unit width with continuously inhomogeneous layers. The research resulted in the necessary ratios for determining the physical and mechanical characteristics of the equivalent beam's layers according to the structure and material properties of the original beam's phases, and a system of static, geometric and physical relationships for the approximate two-dimensional problem. An example realization of the obtained equations and relations is given. Namely, the completed solution of the problem of static bending a composite console of arbitrary structure with a load on a free end surface taking into account a uniform change of temperature field is presented.

The results of solving the problem of bending the composite console can be directly used to predict the strength and rigidity of such elements. The given system of equations and relations is the starting point for the construction of non-classical deformation models and solving a wide range of problems concerning the deformation of a direct composite beams.

References

[1] Kucher N K, Zarazovskii M N and Danil'chuk E L 2013 Deformation and strength of laminated carbon-fiber-reinforced plastics under a static thermomechanical loading *Mechanics of Composite Materials* **48**(6) pp 669-80

[2] Danil'chuk E L, Kucher N K, Kushnarev A P, Potapov A M, Rudnitskii N P, Samusenko A A and Filatov V E 2015 Deformation and Strength of Unidirectional Carbon-Fiber-Reinforced Plastics at Elevated Temperatures *Strength of Materials* **47**(4) pp 573-8

[3] Neutov S, Sydorchuk M and Surianinov M 2019 Experimental Studies of Reinforced Concrete and Fiber-Reinforced Concrete Beams with Short-Term and Long-Term Loads *Mater. Sci. Forum* **968** pp 227-33

[4] Lekhnitsky S G 1977 *The theory of elasticity of an anisotropic body* (Moscow: Science) 416

[5] Soos E 1963 Sur le problème de Saint-Venant dans le cas des barres hétérogènes avec anisotropie cylindrique *Bull. Math. Soc. Sci. Math. Phys. de la R.P.R.* **55**(7) pp 61-75

[6] Muskhelishvili N I 1966 *Some basic problems of the mathematical theory of elasticity* (Moscow: Science) 708

[7] Koval'chuk S B, Gorik A V, Pavlikov A N and Antonets A V 2019 Solution to the Task of Elastic Axial Compression-Tension of the Composite Multilayered Cylindrical Beam *Strength Mater.* **51**(2) pp 240-51 doi: 10.1007/s11223-019-00070-z

[8] Lekhnitskii S G 1968 *Anisotropic Plate* (New York: Gordon and Breach) 534

[9] Gerstner R W 1968 Stresses in a Composite Cantilever *Journal of Composite Materials* **2**(4) pp 498-501

[10] Cheng S, Wei X and Jiang T 1989 Stress distribution and deformation of adhesive-bonded laminated composite beams *ASCE J. Eng. Mech* **115** pp 1150-62

[11] Zhao L, Chen W Q and Lü C F 2012 New assessment on the Saint-Venant solutions for functionally graded beams *Mechanics Research Communications* **43** pp 1-6

[12] Goryk O V and Kovalchuk S B 2018 Elasticity theory solution of the problem on plane bending of a narrow layered cantilever beam by loads at its end *Mechanics of Composite Materials* **54**(2) pp 179-190 doi:10.1007/s11029-018-9730-z

[13] Jiang A M and Ding H J 2005 The analytical solutions for orthotropic cantilever beams (I): Subjected to surface forces *Journal of Zhejiang University* A **6**(2) pp 126-31

[14] Ding H J, Huang D J and Wang H M 2006 Analytical solution for fixed-fixed anisotropic beam subjected to uniform load *Appl. Math. Mech.* **27**(10) pp 1305-10

[15] Huang D J, Ding H J and Chen W Q 2007 Analytical solution for functionally graded anisotropic cantilever beam under thermal and uniformly distributed load *Journal of Zhejiang University* A **8**(9) pp 1351-55

[16] Zhong Z and Yu T 2007 Analytical solution of a cantilever functionally graded beam *Composites Science and Technology* **67**(3-4) pp 481-88

[17] Wang M and Liu Y 2010 Analytical solution for bi-material beam with graded intermediate layer *Composite Structures* **92** pp 2358-68

[18] Daneshmehr A, Momeni S and Akhloumadi M 2012 Exact elasticity solution for the density functionally gradient beam by using airy stress function *Applied Mechanics and Materials* **110** pp 4669-76

[19] Yang Q, Zheng B, Zhang K and Li J 2014 Elastic solutions of a functionally graded cantilever beam with different modulus in tension and compression under bending loads *Applied Mathematical Modelling* **38**(4) pp 1403-16

[20] Benguediab S, Tounsi A, Abdelaziz H and Meziane M 2017 Elasticity solution for a cantilever beam with exponentially varying properties *Journal of Applied Mechanics and Technical Physics* **58**(2) pp 354-61

[21] Goryk A V and Koval'chuk S B 2018 Solution of a Transverse Plane Bending Problem of a Laminated Cantilever Beam Under the Action of a Normal Uniform Load *Strength of Materials* **50**(3) pp 406-18

[22] Esendemir U, Usal M R and Usal M 2006 The effects of shear on the deflection of simply supported composite beam loaded linearly *J. Reinf. Plast. Compos* **25** pp 835-46
[23] Huang D J, Ding H J and Chen W Q 2007 Analytical solution for functionally graded anisotropic cantilever beam subjected to linearly distributed load *Applied Mathematics and Mechanics* **28**(7) pp 855-60
[24] Daouadji T H, Henni A H, Tounsi A and El Abbes A B 2013 Elasticity Solution of a Cantilever Functionally Graded Beam *Applied Composite Materials* **20**(1) pp 1-15
[25] Koval'chuk S B and Goryk A V 2019 Solution of the problem on an elastic bending of a multilayer narrow beam by normal linear loads on longitudinal faces *International Applied Mechanics* **55**(2)
[26] Pagano N J 1969 Exact solutions for composite laminates in cylindrical bending *J. Compos. Mater.* **3** pp 398-411
[27] Sankar B V 2001 An elasticity solution for functionally graded beams *J. Composites Science and Technology* **61**(5) pp 689-96
[28] Silverman I K 1964 Orthotropic beams under polynomial loads *ASCE Journal of the Engineering Mechanics Division* **90** pp 293-319
[29] Hashin Z 1967 Plane anisotropic beams *J. Appl. Mech*. **34**(2) pp 257-62
[30] Jiang A M and Ding H J 2005 The analytical solutions for orthotropic cantilever beams (II): solutions for density functionally graded beams *Journal of Zhejiang University* A **6**(3) pp 155-8
[31] Ding H J, Huang D J and Chen W Q 2007 Elasticity solutions for plane anisotropic functionally graded beams *International Journal of Solids and Structures* **44**(1) pp 176-196
[32] Huang D J, Ding H J and Chen W Q 2009 Analytical solution and semi-analytical solution for anisotropic functionally graded beam subject to arbitrary loading // *Science in China Series G: Physics, Mechanics and Astronomy* **52**(8) pp 1244-56
[33] Nie G J, Zhong Z and Chen S 2013 Analytical solution for a functionally graded beam with arbitrary graded material properties *Composites* B **44** pp 274-82
[34] Zhang L, Gao P and Li D 2012 New Methodology to Obtain Exact Solutions of Orthotropic Plane Beam Subjected to Arbitrary Loads *Journal of Engineering Mechanics* **138**(11) pp 1348-56
[35] Koval'chuk S B 2020 Exact Solution of the Problem on Elastic Bending of the Segment of a Narrow Multilayer Beam by an Arbitrary Normal Load *Mech. Compos. Mater*. **56**(1) pp 55-74
[36] Thurnherr C, Groh R M J, Ermanni P and Weaver P M 2016 Higher-order beam model for stress predictions in curved beams made from anisotropic materials *International Journal of Solids and Structures* **97-98** pp 16–28 doi: 10.1016/j.ijsolstr.2016.08.004
[37] Sayyad A S and Ghugal Y M 2017 Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature *Composite Structures* **171** pp 486-504
[38] Piskunov V G, Goryk A V and Cherednikov V N 2000 Modeling of transverse shears of piecewise homogeneous composite bars using an iterative process with account of tangential loads. 1. Construction of a model *Mechanics of Composite Materials* **36**(4) pp 287-96
[39] Shvabyuk V I, Rotko S V and Uzhegova O A 2014 Bending of a Composite Beam with a Longitudinal Section *Strenght of Materials* **46**(4) pp 558-66